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Bijections and recurrences for integer partitions into a bounded number of parts

Marilena Barnabei *, Flavio Bonetti, Matteo Silimbani

Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato, 5 40127 Bologna, Italy

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ABSTRACT

Let $\Pi_s(n)$ denote the set of partitions of the integer n into exactly s parts, and $\Pi_s^{(2)}(n)$ the subset of $\Pi_s(n)$ containing all partitions whose two largest parts coincide. We present a bijection between $\Pi_s^{(2)}(n)$ and $\Pi_{s-1}(m)$ for a suitable $m < n$ in the cases $s = 3, 4$. Such bijections yield recurrence formulas for the numbers $P_3(n)$ and $P_4(n)$ of partitions of n into 3 and 4 parts. Furthermore, we show that the present approach can be extended to the case $s = 5, 6$.

1. Introduction

A *partition* π of an integer n is a way of expressing n as a sum $n = h_1 + h_2 + \cdots + h_s$ of positive integers satisfying the condition $h_1 \geq h_2 \geq \cdots \geq h_s$. The integers h_i are called *parts* of the partition. We denote by $P(n)$ the number of partitions of the integer n and by $P_s(n)$ the number of partitions of n into s parts.

The problem of counting partitions of an integer n was first undertaken by Euler [4], who found the generating function for the numbers $P(n)$. Since then, integer partitions have been deeply studied by mathematicians for their fascinating properties. Most of the present abundant literature deals with recurrence formulas, identities and asymptotics (see [2,6] and [9] for extensive surveys on the subject). In the 80's, the number of partitions of an integer was studied according to the number s of its parts. Some explicit and recursive formulas for the number $P_s(n)$ of partitions of n into s parts have been found for small values of s (see [3], and sequences A001399, A001400 and A001401 in [7]). This topic was first investigated by Andrews [1], who discovered a relationship between the number of partitions into three parts and the number of triangles with integer sides. Further results in this direction can be found in [5].

In this work, we give a new contribution to the subject, by exhibiting two bijections between suitable sets of partitions of an integer into a bounded number of parts. These bijections allow us to find some recurrence formulas for the numbers of partitions of n into three or four parts. To this end, we focus on the subset of twin partitions, namely, partitions of n whose two greatest parts coincide. First of all, we remark that the set of partitions of $n - 1$ into s parts corresponds bijectively to the set of non-twin partitions of n. This implies that the sequence $P_s^{(2)}(n)$ of the number of twin partitions of n into s parts is the image under the backward difference operator Δ of the sequence $P_s(n)$. As a consequence, we obtain a recurrence formula for the sequence $P_s(n)$ as soon as we find a bijection between the set of twin partitions of n into s parts and some set of partitions of a suitable integer into a smaller number of parts. In Sections 3 and 4, we exhibit explicit bijections for the cases $s = 3, 4$. With similar arguments, we succeed in finding some recurrence relations for the number of twin partitions of n into five and six parts.

* Corresponding author.
E-mail addresses: barnabei@dm.unibo.it (M. Barnabei), bonetti@dm.unibo.it (F. Bonetti), silimban@dm.unibo.it (M. Silimbani).

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2. Preliminaries

Let n be a positive integer. A partition π of n is a weakly decreasing sequence of positive integers $h_1, h_2, \ldots, h_s$ such that $n = h_1 + h_2 + \cdots + h_s$. If n is known, we can omit the integer h_1, since it can be recovered as $h_1 = n - \sum_{i=2}^{s} h_s$. We say that the list $(h_2, \ldots, h_s)$ *encodes* the partition π, and that $(h_2, \cdots, h_s)$ is the *reduced representation* of π. Conversely, a non-increasing list $(h_2, \ldots, h_s)$ of positive integers encodes a partition of n whenever $n - \sum_{i=2}^{s} h_s \geq h_2$.

We will also represent a partition $n = h_1 + \cdots + h_s$ by the left-up-justified Ferrers diagram on n boxes whose j-th column contains h_j cells.

The set of partitions of the integer n into exactly s parts will be denoted by $\Pi_s(n)$. We denote by $P_s(n)$ the cardinality of $\Pi_s(n)$. It is immediately seen that $P_s(n)$ is also the number of partitions of the integer $n-s$ into at most s parts by suppressing 1 in each part.

The recurrence formulas for the sequence $P_s(n)$ in the present work are based on the following result:

Proposition 1. *For every $n > s$, the set of partitions of n into s parts whose two largest parts do not coincide corresponds bijectively to the set $\Pi_s(n-1)$.*

Proof. The assertion follows immediately from the fact that a list $(h_2, \ldots, h_s)$ such that $n - \sum_{i=2}^{s} h_i > h_2$ is the reduced representation of both a partition of n into s parts whose first two parts do not coincide and a partition of $n-1$ into s parts. □

For example, the 4-tuple $(8, 6, 4, 1)$ is the reduced representation of both the partitions $29 = 10 + 8 + 6 + 4 + 1$ and $28 = 9 + 8 + 6 + 4 + 1$.

We say that $n = h_1 + h_2 + \cdots + h_s$ is a *twin partition* of n if $h_1 = h_2$. We denote by $\Pi_s^{(2)}(n)$ the set of twin partitions in $\Pi_s(n)$, and by $P_s^{(2)}(n)$ the cardinality of $\Pi_s^{(2)}(n)$.

The statement of Proposition 1 can be reformulated in terms of twin partitions as follows:

$$P_s(n) - P_s(n-1) = P_s^{(2)}(n). \tag{1}$$

Repeated applications of identity (1) yield:

Proposition 2. *For every $n \geq s$, we have*

$$P_s(n) = \sum_{k=s}^{n} P_s^{(2)}(k). \quad \square$$

Proposition 1 yields a bijective proof of identity (1). This identity can be proved by a generating function argumentations as well.

Recall that the generating function $f_s(x)$ of the sequence $P_s(n)$ is

$$f_s(x) = x^s \prod_{i=1}^{s} \frac{1}{1-x^i} = \frac{x^s}{(1-x)(1-x^2)\cdots(1-x^s)}.$$

Similarly, the generating function $f_s^{(2)}(x)$ of the sequence $P_s^{(2)}(n)$ is (see e.g. [8])

$$f_s^{(2)}(x) = x^s \cdot \prod_{i=2}^{s} \frac{1}{1-x^i} = \frac{x^s}{(1-x^2)(1-x^3)\cdots(1-x^s)}.$$

Then, identity (1) can be obtained by comparing the generating functions $f_s(x)$ and $f_s^{(2)}(x)$.

3. Partitions into three parts

In this section we state some recurrence relations involving the sequence $P_3(n)$. Our starting point is the following observation: a twin partition $\pi \in \Pi_3^{(2)}(n)$ can be recovered from a single parameter, namely, the size k of its smallest part. Since integer partitions into two parts can be characterized by a single parameter as well, it seems reasonable to look for a bijection between twin partitions of n into three parts and partitions into two parts of a smaller integer. We start by defining the bijection when n is a multiple of 3.

For every integer $n \equiv 0 \pmod 3$, we define a map

$$\varphi_0 : \Pi_3^{(2)}(n) \to \Pi_2\left(\frac{n}{3} + 1\right)$$

in terms of reduced representations as follows:

$$\varphi_0\left(\left(\frac{n-k}{2}, k\right)\right) = \left(\frac{n-3k}{6} + 1\right).$$

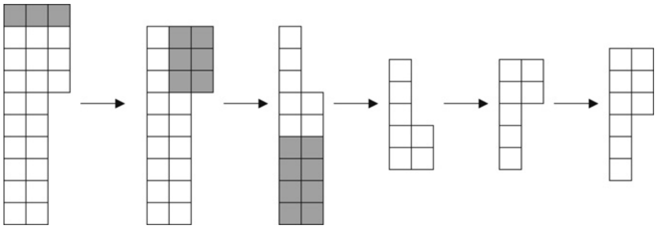

Fig. 1. The bijection φ_0 maps the twin partition $24 = 10 + 10 + 4$ into the partition $9 = 6 + 3$.

In other words, the twin partition $n = \frac{n-k}{2} + \frac{n-k}{2} + k$ is mapped into the partition $\frac{n}{3} + 1 = \frac{n+3k}{6} + \frac{n-3k+6}{6}$. Since $n - 3k$ must be even, the number $\frac{n-3k}{6}$ is an integer, and the map φ_0 is well defined. We have the following:

Theorem 3. *The map* $\varphi_0 : \Pi_3^{(2)}(n) \to \Pi_2\left(\frac{n}{3} + 1\right)$ *is a bijection for every* $n \equiv 0 \pmod 3$.

Proof. It is immediate that φ_0 is injective. The bijectivity of φ_0 is proved as soon as we show that the two sets $\Pi_3^{(2)}(n)$ and $\Pi_2\left(\frac{n}{3} + 1\right)$ have the same cardinality. In fact, $(\frac{n-k}{2}, k)$ is the reduced representation of a partition in $\Pi_3^{(2)}(n)$ whenever $\frac{n-k}{2} \geq k$. Hence, we have

$$P_3^{(2)}(n) = \left|\left\{k : 1 \leq k \leq \frac{n}{3}, n - k \equiv 0 \ (\mathrm{mod}\ 2)\right\}\right|.$$

On the other hand, we have

$$P_2\left(\frac{n}{3} + 1\right) = \left|\left\{h : 2 \leq 2h \leq \frac{n}{3} + 1\right\}\right|,$$

as desired. $\square$

The bijection defined above can be interpreted in terms of Ferrers diagrams. Given a twin partition $\pi \in \Pi_3^{(2)}(n)$, with $n \equiv 0 \pmod 3$, and reduced representation $(\frac{n-k}{2}, k)$, the partition $\varphi_0(\pi)$ is obtained by performing the following operations on the Ferrers diagram F associated with π, where we indicate in parentheses the number of boxes of the diagram obtained at each step:

- delete the topmost row of F ($n - 3$ boxes);
- delete the last two boxes in each other row of length 3 ($n - 1 - 2k$);
- delete $\frac{n}{3} - k$ rows of length 2 in F ($\frac{n}{3} - 1$);
- rearrange the new rows in F in non-increasing order ($\frac{n}{3} - 1$);
- add a row of length 2 atop the diagram, getting the Ferrers shape of $\varphi_0(\pi)$ ($\frac{n}{3} + 1$).

An example of such a bijection is given in Fig. 1.

The bijection φ_0 is defined only when $n \equiv 0 \pmod 3$. The remaining cases can be handled by noticing that a twin partition of $n \not\equiv 0 \pmod 3$ can be uniquely mapped into a twin partition of a suitable $n' \equiv 0 \pmod 3$:

- if $n = 3m + 1$, we define the bijection $\xi_1 : \Pi_3^{(2)}(3m + 1) \to \Pi_3^{(2)}(3m - 3)$ as follows: for every $\pi \in \Pi_3^{(2)}(3m + 1)$ whose reduced representation is (h_2, h_3), $\xi_1(\pi)$ is the partition encoded by the list $(h_2 - 2, h_3)$, and the composition $\varphi_1 = \varphi_0 \circ \xi_1 : \Pi_3^{(2)}(3m + 1) \to \Pi_2(m)$ is the desired bijection;
- if $n = 3m + 2$, we define the bijection $\xi_2 : \Pi_3^{(2)}(3m + 2) \to \Pi_3^{(2)}(3m)$ as follows: for every $\pi \in \Pi_3^{(2)}(3m + 2)$ whose reduced representation is (h_2, h_3), $\xi_2(\pi)$ is the partition encoded by the list $(h_2 - 1, h_3)$, and the composition $\varphi_2 = \varphi_0 \circ \xi_2 : \Pi_3^{(2)}(3m + 2) \to \Pi_2(m + 1)$ is the desired bijection.

The bijections φ_0, φ_1 and φ_2 give a combinatorial interpretation of the following recursion for the sequence $P_3(n)$:

Theorem 4. *The sequence* $(P_3(n))_{n\geq 3}$ *satisfies the following recurrence formula:*

$$P_3(n) = P_3(n - 1) + P_2(\psi(n)), \tag{2}$$

where

$$\psi(n) = \begin{cases} \dfrac{n}{3} + 1 & \text{if } n \equiv 0 \bmod 3 \\ \dfrac{n-1}{3} & \text{if } n \equiv 1 \bmod 3 \\ \dfrac{n+1}{3} & \text{if } n \equiv 2 \bmod 3. \quad \square \end{cases}$$

For the sake of completeness, we state a further recurrence formula for $P_3(n)$, which yields an explicit formula for $P_3(n)$. This recurrence can be obtained from the observation that a pair $(m, n-2m)$ encodes a partition $\pi \in \Pi_3^{(2)}(n)$ whenever

$$\left\lfloor \frac{n+2}{3} \right\rfloor \le m \le \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Hence,

$$P_3^{(2)}(n) = \left\lfloor \frac{n+1}{2} \right\rfloor - \left\lfloor \frac{n+2}{3} \right\rfloor, \tag{3}$$

which gives the following identities for the integers $P_3(n)$:

Proposition 5. *For every $n \ge 3$, the integers $P_3(n)$ satisfy the recurrence*

$$P_3(n) = P_3(n-1) + \left\lfloor \frac{n+1}{2} \right\rfloor - \left\lfloor \frac{n+2}{3} \right\rfloor. \quad \square \tag{4}$$

Proposition 6. *For every $n \ge 3$, we have*

$$P_3(n) = \left\lfloor \frac{n}{2} \right\rfloor^2 - E(n)\frac{n}{2} - \frac{3}{2}\left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor - 1 \right) - \left\lfloor \frac{n}{3} \right\rfloor [n]_3,$$

where $[n]_3$ is the congruence class of n modulo 3 and

$$E(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. In Proposition 2 we proved that

$$P_3(n) = \sum_{k=3}^{n} P_3^{(2)}(k).$$

Exploiting the explicit formula (3) we get the assertion. $\square$

Identity (3) shows that the integers $P_3^{(2)}(n)$ and $P_3^{(2)}(n-6)$ differ only by 1, i.e.,

$$P_3^{(2)}(n) = 1 + P_3^{(2)}(n-6).$$

This last identity together with Proposition 1 gives the following:

Proposition 7. *The sequence $P_3(n)$ satisfies the following recurrence:*

$$P_3(n) = 1 + P_3(n-1) + P_3(n-6) - P_3(n-7),$$

where we set $P_3(h) = 0$ if $h \le 0$. $\square$

4. Partitions into four parts

We extend the approach of the previous section to partitions of n into four parts by deriving a recurrence relation for the sequence $P_4(n)$.

A twin partition $\pi \in \Pi_4^{(2)}(n)$ is determined by two parameters, namely, the sizes k and h of its last two parts. Note that the difference between n and $k+h$ must be even. This observation suggests looking for a bijective map between the sets $\Pi_4^{(2)}(n)$ and $\Pi_3(t)$, for some $t < n$.

We start with the case $n \equiv 0 \pmod 2$. Given an even integer n, we define the map

$$\eta_0 : \Pi_4^{(2)}(n) \to \Pi_3\left(\frac{n}{2} + 1\right)$$

in terms of reduced representations:

$$\eta_0\left(\left(\frac{n-k-h}{2}, k, h\right)\right) = \left(\frac{k+h}{2}, \frac{k-h+2}{2}\right).$$

Since $k+h$ is even, the map is well defined and we have:

Theorem 8. *For every even number n the function η_0 is a bijection between the sets $\Pi_4^{(2)}(n)$ and $\Pi_3\left(\frac{n}{2}+1\right)$.*

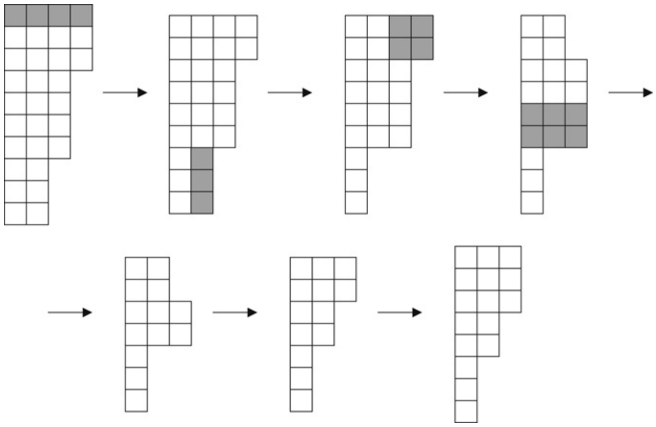

Fig. 2. The bijection η_0 maps the twin partition $30 = 10 + 10 + 7 + 3$ into the partition $16 = 8 + 5 + 3$.

Proof. The map η_0 is obviously injective. It only remains to prove that $P_4^{(2)}(n) = P_3(\frac{n}{2} + 1)$. In fact, the triple $(\frac{n-k-h}{2}, k, h)$ encodes a twin partition of n into four parts whenever the first two parts of π are greater than or equal to k. This implies that

$$P_4^{(2)}(n) = |\{(k, h) : 1 \le h \le k, h + k \equiv 0 \pmod 2,\ 3k + h \le n\}| .$$

Similarly, we have

$$P_3\left(\frac{n}{2} + 1\right) = \left|\left\{(a, b) : 1 \le b \le a, 2a + b \le \frac{n}{2} + 1\right\}\right| .$$

The assertion is proved by setting $a = \frac{k+h}{2}$ and $b = \frac{k-h}{2} + 1$. $\square$

As before, we describe the action of the bijection η_0 on the Ferrers diagrams. If π has reduced representation $(\frac{n-h-k}{2}, k, h)$, the diagram of the partition $\eta_0(\pi)$ is obtained as follows:

- delete the topmost row of the Ferrers diagram F associated with π ($n - 4$ boxes);
- delete the rightmost box in every row of F with exactly two cells ($\frac{n+3k+h}{2} - 4$);
- delete the last two boxes in each other row of length 4 ($\frac{n+3k-3h}{2} - 2$);
- if $q = 2s$ is the number of rows of cardinality 3, delete s of them ($\frac{n}{2} - 2$);
- rearrange the new rows in F in non-increasing order ($\frac{n}{2} - 2$);
- add a row of length 3 atop the diagram, getting the Ferrers shape of $\eta_0(\pi)$ ($\frac{n}{2} + 1$).

Fig. 2 shows an example of this bijection.

Up to now, the bijection η_0 has been defined only in the case of even integers. We extend the bijection to the odd case as follows: if $n = 2m + 3$, we define a bijection

$$\beta : \Pi_4^{(2)}(2m + 3) \to \Pi_4^{(2)}(2m)$$

in terms of reduced representations, as follows:

$$(h_2, h_3, h_4) \mapsto (h_2 - 1, h_3 - 1, h_4). \tag{5}$$

Note that the triple $(h_2 - 1, h_3 - 1, h_4)$ encodes a twin partition of the integer $2m$, since n odd implies that $h_3 \neq h_4$, namely $h_3 - 1 \ge h_4$. Then, the composition

$$\eta_1 = \eta_0 \circ \beta : \Pi_4^{(2)}(2m + 3) \to \Pi_3(m + 1)$$

is a bijection. The above defined bijections yield a recurrence formula for $P_4(n)$ involving $P_4(n - 1)$ and $P_3(m)$ for some $m < n$:

Proposition 9. *The sequence $P_4(n)$ satisfies the following recurrence formula:*

$$P_4(n) = P_4(n - 1) + P_3(\chi(n)), \quad n \ge 5 \tag{6}$$

where

$$\chi(n) = \begin{cases} \dfrac{n}{2} + 1 & \text{if } n \text{ is even} \\ \dfrac{n-1}{2} & \text{otherwise.} \end{cases} \quad \square$$

Note that the bijection β defined above yields the following identity, when n is an odd integer:

$$P_4^{(2)}(n) = P_4^{(2)}(n-3).$$

The recurrence formula (6) for the sequence $P_4(n)$ is surprisingly simpler than the analogous recurrence for $P_3(n)$. In terms of generating functions, recast as the following equalities:

$$f_3^{(2)}(x) = \frac{1+x^2+x^4}{x^3} f_2(x^3),$$
$$f_4^{(2)}(x) = \frac{1+x^3}{x^2} f_3(x^2).$$

Even though these relations give no explicit information about the bijections φ_i and η_i, they suggest the choice of the codomain of the two maps.

The previous arguments can be in principle extended to the case $s > 4$. However, in the general case, the relations between $f_s^{(2)}(x)$ and $f_{s-1}(x)$ are much more complicated than in the cases $s = 3, 4$, and it seems that they do not suggest natural bijections.

5. Further recurrence formulas

The approach introduced in the previous section can be extended to the general case of the set $\Pi_s^{(t)}(n)$ of partitions of n into s parts whose t first parts coincide. First of all, an analog of the bijection β described in the previous section can be defined as follows:

$$\beta_s^{(t)} : \Pi_s^{(t)}(n-t) \to \Pi_s^{(t)}(n)$$
$$\beta_s^{(t)}(\sigma) = \tau,$$

where σ is the partition $n - t = h_1 + \cdots + h_s$ and τ is the partition $n = (h_1 + 1) + \cdots + (h_t + 1) + h_{t+1} + \cdots + h_s$.

It is easy to see that the function $\beta_s^{(t)}$ is injective and its image set is $\Pi_s^{(t)}(n) - \Pi_s^{(t+1)}(n)$. Hence, denoting by $P_s^{(t)}$ the cardinality of the set $\Pi_s^{(t)}$ we get

$$P_s^{(t+1)}(n) = P_s^{(t)}(n) - P_s^{(t)}(n-t). \tag{7}$$

On the other hand, the integer $P_s^{(s-1)}(n)$ can be easily computed by considering the possible values of the last part:

$$P_s^{(s-1)}(n) = \left\lfloor \frac{n+s-2}{s-1} \right\rfloor - \left\lfloor \frac{n+s-1}{s} \right\rfloor. \tag{8}$$

Identity (8) yields immediately recurrence formulas for both $P_s^{(s-1)}(n)$ and $P_s^{(s-2)}(n)$:

$$P_s^{(s-1)}(n) = P_s^{(s-1)}(n - s^2 + s) + 1. \tag{9}$$

$$P_s^{(s-2)}(n) = P_s^{(s-2)}(n - s + 2) + \left\lfloor \frac{n+s-2}{s-1} \right\rfloor - \left\lfloor \frac{n+s-1}{s} \right\rfloor. \tag{10}$$

For example, in the case $n = 4$, identity (10) turns into a recurrence formula for the number of twin partitions of n into four parts, namely

$$P_4^{(2)}(n) = P_4^{(2)}(n-2) + \left\lfloor \frac{n+2}{3} \right\rfloor - \left\lfloor \frac{n+3}{4} \right\rfloor,$$

which holds for every integer $n \geq 4$.

Consider now the sequence $P_s^{(t)}(n)$. It is easy to verify that, for every $t \leq s$, we have

$$P_s^{(t)}(n) = \sum_{h \geq 0} P_{s-1}^{(t)}(n - 1 - sh),$$

which becomes, in the case $s = 5, t = 2$,

$$P_5^{(2)}(n) = \sum_{h \geq 0} P_4^{(2)}(n - 1 - 5h). \tag{11}$$

The relationship between $P_4^{(2)}(n)$ and $P_3(n)$ can be exploited in order to deduce some explicit formulas for the number of twin partitions of n into a greater number of parts from identity (11). For example, we can express the integers $P_5^{(2)}(n)$ in terms of the integers $P_3(m)$, for some suitable m:

$$P_5^{(2)}(n) = \begin{cases} P_3\left(\frac{n+1}{2}\right) + \sum_{h=0}^{\lfloor \frac{n-13}{10} \rfloor} \left(P_3\left(\frac{n-7}{2} - 5h\right) + P_3\left(\frac{n-9}{2} - 5h\right)\right) & \text{if } n \text{ is odd} \\ \sum_{h=0}^{\lfloor \frac{n-8}{10} \rfloor} \left(P_3\left(\frac{n-2}{2} - 5h\right) + P_3\left(\frac{n-4}{2} - 5h\right)\right) & \text{if } n \text{ is even.} \end{cases}$$

Similarly, we can derive an expression for the number of twin partitions of n into six parts:

$$P_6^{(2)}(n) = \begin{cases} \sum_{h\geq 0} \sum_{\substack{k\not\equiv 0 \pmod 3 \\ k>0}} P_3\left(\frac{n-1}{2} - k - 5h\right) & \text{if } n \text{ is odd} \\ \sum_{h\geq 0} P_3\left(\frac{n}{2} - 3h\right) + \sum_{h\geq 0} \sum_{\substack{k\not\equiv 0 \pmod 3 \\ k>0}} P_3\left(\frac{n-6}{2} - k - 5h\right) & \text{if } n \text{ is even.} \end{cases}$$

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